Dynamics of coupled gap solitons

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We show analytically that solitonic states of two modes associated with different gap edges are formed when incident waves are sent in layered dielectric systems with a small band gap. The resulting coupled gap solitons are described through a nonlinear Schrödinger equation and propagate with a speed equal to half the speed of light. We discuss the conditions for the existence of coupled gap solitons.

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I. INTRODUCTION

Periodic optical structures attract increasing attention due to their remarkable properties such as the appearance of stop gaps in the spectrum, where the propagation of the linear waves is impossible [1]. Furthermore, periodicity can produce group velocity dispersion (GVD) at the edges of the gaps, even if the composite materials are nondispersive. Combining nonlinearity with such a dispersion may lead to the creation and propagation of solitons. This possibility has been predicted by Winful [2] and discovered numerically by Chen and Mills [3]. In a series of publications [4–10] the theory of the gap solitons has been developed. In particular, it has been shown that when nonlinearity is present, the propagation of light in the gap is possible and occurs in the form of pulses localized in space that are named gap solitons.

From rigorous expansion methods developed in Refs. [5–7] we know that these pulses are governed by the nonlinear Schrödinger (NLS) equation when the incident wave has frequency equal (or close) to that of the gap edge [5,6] or by a system of coupled NLS-like equations if either the frequency is inside a gap or generation of the third harmonic is taken into account [7]. The method of Refs. [5–7] uses Bloch functions as a background of the expansion that makes it rather general; it has been applied so far only to study situations when at nonvanishing periodic modulation of the linear dielectric permittivity either the incident radiation is close to the band edge frequency or the gap width is large enough (a system of the coupled NLS-like equations describing the last case is derived in [7]). Peyraud and Coste [11], on the other hand, used numerical experiments and discovered some interesting propagating features of nonlinear layered media. More specifically, they found that (i) stationary gap solitons are generically unstable (i.e., asymmetry of the system results in the *propagation* of the solitons), and (ii) if the intensity of the applied radiation is strong enough, i.e., is above some critical value the ‘‘trivial solution’’ (according to Ref. [11]) becomes unstable while at the same time, gap solitons appear that propagate with a velocity close to $c/2$. For the present work it is important to mention that the above phenomena are observed when the incident wave has frequency equal to the middle of the gap and the critical intensity is estimated to be of the order of the half of the gap width (in dimensionless units). Physically, this corresponds to a situation where both gap edges play an equivalent role in the wave creation and hence excluded from the conditions of the work presented in Refs. [5–7].

The situation considered in [11] naturally implies a coupling of edges of the gap. This coupling is different from the one considered in [8–10], since in the last case structures with weak modulation depths were examined, while the modulation in [11] is extremely strong. In the case of weak modulation [8–10] coupling occurrs between counterpropagating plane waves with wave numbers determined by the Bragg condition. By analogy, it is reasonable to assume that in the conditions of the numerical experiment of Ref. [11] there is a coupling of standing waves that are Bloch functions of states near the stop gap.

The goal then of the present paper is to investigate in more detail the role of the gap edges in the dynamics and give an analytical explanation for the numerical findings of Peyraud and Coste. It will be shown that the method of de Sterke and Sipe [6,7] can be generalized to take into account *coupling of modes*. Here we mean sufficiently strong coupling, which occurs on a *linear* level of the perturbative expansion, rather than due to the well known cross-phase modulation [14]. It is to be mentioned that another manifestation of mode coupling due to anysotropy of the medium has recently been reported in [12]. In that case weak anisotropy resulted in splitting the frequency (located near an edge of a stop gap large enough) and creating vector solitons of two orthogonal polarizations of the field. The coupling studied in the present paper has a different physical nature: it is due to *two* modes corresponding to *two different edges of a narrow gap*. The medium will be considered isotropic. This will lead to a *single* Schrödinger equation so that respective solitons will have ‘‘scalar’’ character as long as only one TE polarization of the electric field is under consideration.

The structure of the paper is the following: In Sec. II we

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give a precise definition of the problem and describe the analytical technique to be used. In Sec. III we discuss the properties of the coupled gap solitons and in Sec. IV we conclude.

II. MULTIPLE SCALE EXPANSION

Before deriving analytical expressions, let us first discuss some physical and mathematical aspects of the problem. Consider an incident wave having a frequency equal to the frequency at the middle of the stop gap and assume that the latter is narrow enough (the arguments that follow qualitatively apply also for the case of large radiation intensity). We denote by ω_1 and ω_2 the lower and upper gap boundaries, respectively, i.e., assume $\omega_1 < \omega_2$. In the presence of nonlinearity, an incident wave should cause generation of solitons at both edges. Since soliton pulses are products of a delicate balance between GVD and nonlinearity a small ''distance'' between the band edges causes *simultaneous* excitation of solitons on one hand, whereas, on the other hand, in order for both solitons to exist, the balance between nonlinearity and dispersion in *both* edges has to be provided. The last requirement is not evident or trivial. Indeed, the sign of GVD at the edges of one of the gaps is different. Thus, in order for gap solitons to exist, it is necessary to have another source of GVD.

In order to give an idea of the possible origin of the induced GVD, let us recall the well known fact of the linear theory [15]: If there is a cross point of two branches of a spectrum of a system (i.e., a degeneration point), then a small perturbation removes the degeneracy and results in two new branches. On the other hand, this means that in leading order there are two representations of a linear wave (in the nonlinear theory it serves as a background for an envelope soliton). They are either perturbed (i.e., having the gap) or unperturbed linear combinations. Now assume that the coupling of two perturbed modes results exctly in one of the unperturbed states, then the induced GVD will be associated with the unperturbed state. As shown below, it is this scenario that occurs due to coupling of modes corresponding to different edges. Moreover, the linear Bloch functions associated with the edges of the same gap have different parity, while the effective nonlinearity is formed just by these functions [6]. We will see that this leads to additional requirements for the problem parameters and in a number of cases narrows possible types of solutions.

The mathematical peculiarity of the above statements is that the difference $\omega_2 - \omega_1$ itself is now a small parameter. This fact has to be taken into account at the *first* stage of the expansion developed in [6] in order to hold the method self-consistent. In other words, let us consider the nonlinear equation for the real electric field E:

$$-c^2 \frac{\partial^2 E}{\partial x^2} + \epsilon(x) \frac{\partial^2 E}{\partial t^2} = -4\pi\chi^{(3)}(x) \frac{\partial^2}{\partial t^2} E^3, \tag{1}$$

where $\epsilon(x)$ is the periodic dielectric permittivity and $\chi^{(3)}(x)$ the periodic nonlinear susceptibility [hereafter both $\epsilon(x)$ and $\chi^{(3)}(x)$ are assumed to have the same period] and c is the speed of light. We expand the electric field in the form

$$E = \mu e^{(1)} + \mu^2 e^{(2)} + \cdots, \tag{2}$$

where μ is a small parameter $\mu \ll 1$, and use the substitution

$$e^{(1)}(x,t) = [a_1(x,t)\phi_1(x) + a_2(x,t)\phi_2(x)]e^{-i\omega t} + \text{c.c.} \tag{3}$$

Hereafter the indices 1 and 2 refer to the lower and upper edges of the gap, respectively, $\phi_m(x)$ is the Bloch function associated with the edge m, $\omega = (\omega_1 + \omega_2)/2$ is the frequency corresponding to the middle of the gap, and c.c. means complex conjugation. In the representation of Eq. (3) we have imposed

$$\frac{\omega_2 - \omega_1}{\omega} = \mu^j, \quad j \geqslant 2 \tag{4}$$

and hence the fact that the amplitudes $a_m(x,t)$ are slowly varying *has been taken into account*. Note that the multiscale expansion

$$t = t_0 + \mu t_1 + \mu^2 t_2 + \cdots, \quad x = x_0 + \mu x_1 + \mu^2 x_2 + \cdots$$

results in $(\omega_2 - \omega_1)t = \omega t_j$ and a_m depends on x_j and t_j with $j \geqslant 1$: $a_m = a_m(x_1, x_2, \ldots; t_1, t_2, \ldots)$ (see [6], or the development below). Thus, comparing (2) and (4), the physical condition for the expansion to be valid can be expressed through the relation

$$\frac{\omega_2 - \omega_1}{\omega_2 + \omega_1} \sim \left(\frac{\chi^{(3)} I}{\epsilon} \right)^{j/2}. \tag{5}$$

Equation (4) [or (5)] can be considered a mathematical definition of the small parameter. In this context the cases $j \geqslant 3$ and $j = 2$ can be interpreted as the cases of relatively (i.e., compared with the gap width) large and small amplitudes. They correspond to distinct physical situations and will be considered separately in what follows.

In a realistic situation the value of the right hand side of Eq. (5) can be of order of 10^{-3}. On the other hand, the necessary difference $\omega_2 - \omega_1$ can be reached even for the second gap of the layered structure (for instance, in the region where the gap collapses, as in the analysis of particular structures [13]).

The condition of Eq. (5) has a transparent physical meaning. In order to allow for soliton propagation, the effective GVD induced by coupling must be much stronger than the GVD at the gap edges. Nonlinearity must be of the same order with the effective dispersion. When GVD at the edges is increasing inversly proportional to the gap width, the latter must be comparable with the intensity of the pulse. The effective nonlinearity, however, is formed through the Bloch functions. Hence, the balance between nonlinearity and dispersion depends not only on $\chi^{(3)}$ but also on the ''linear'' properties of the periodic medium. This is the reason that depending on the properties of the system, localized coupled pulses can be created either at $j = 2$ or at $j \geqslant 3$.

III. PROPERTIES OF THE COUPLED GAP SOLITONS

There is no need to discuss here all the details of the multiscale method that are represented in the original papers [6,7]. Instead we emphasize only some peculiarities caused by the specific form of Eq. (3). In order to obtain the equation for slowly varying amplitude one has to substitute the expansion of Eq. (2) for the electric field in Eq. (1) and collect the terms of different powers of μ. Evidently, Eq. (3) under these assumptions solves the problem to first order. For the second order term that is proportional to μ^2 (in Ref. [6] it is the called companion term while Eq. (3) is referred to as the principal one) we write

$$e^{(2)}=\sum_l b_l(x_1,x_2,\ldots;t_1,t_2,\ldots)\phi_l(x_0)e^{-i\omega t_0}+\text{c.c.} \tag{6}$$

Also, we recall the definition for the inner product

$$\langle l|f(x)|m\rangle\equiv\frac{1}{L}\int_0^L\overline{\phi}_l(x)f(x)\phi_m(x)dx \tag{7}$$

(the bar stands for the complex conjugation and L is the length of the periodic stack) and the following additional properties: (i) $\text{Im}\phi_1(x)=\text{Im}\phi_2(x)=0$ since we are considering gap edges. In other words, $\phi_1(x)$ and $\phi_2(x)$ are standing waves. (ii) $\langle l|\epsilon|m\rangle=\delta_{l,m}$, with $\delta_{l,m}$ being the Kroneker δ; this is the normalization condition for the states. (iii) $\langle 1|2\rangle=\langle 1|\partial/\partial x_0|1\rangle=\langle 2|\partial/\partial x_0|2\rangle=0$; these follow from the fact that $\phi_1(x)$ and $\phi_2(x)$ have different parity (each of them is either even or odd).

Upon multiplication of Eq. (1) sequentially by $\langle 1|$ and $\langle 2|$ and considering the companion terms we arrive at the system of equations for a_1 and a_2:

$$\frac{\partial a_1}{\partial t_1}-iv_{\text{st}}\frac{\partial a_2}{\partial x_1}=0,\quad \frac{\partial a_2}{\partial t_1}+iv_{\text{st}}\frac{\partial a_1}{\partial x_1}=0, \tag{8}$$

where

$$v_{\text{st}}=\frac{c^2}{\omega}\langle 1|\frac{\partial}{\partial x_0}|2\rangle. \tag{9}$$

There are few important consequences of the system of Eq. (8). First, Eq. (8) has a solution characterized by

$$a_1=\pm ia_2. \tag{10}$$

This means that the excitations at edges are coupled. Their phases have an initial phase shift $\pm\pi/2$. The related functions a_j depend on t_1 and x_1 only through the combination $z=x_1\pm v_{\text{st}}t_1$. In other words, a_j represents a wave packet traveling with the velocity v_{st}. Thus the coupled gap solitons carry the field energy at the frequency incide the gap. For the sake of definitness in what follows we consider the wave with the polarization $a_2=ia_1$ (respectively it will be $z=x_1-v_{\text{st}}t_1$).

In the generic case v_{st} is not a constant but depends on k. Taking into account the qualitative arguments of the previous section, Eq. (10) and the development of the Appendix one concludes that it is this mechanism that provides us with the new source of GVD. The value of the velocity v_{st} can be easily estimated in the cases when the general form of the normalized Bloch function $\phi_1(x)$ that is bordering the stop gap is approximated by $n_1\cos(k_Bx+\phi_0)$, $k_B=\pi/d$ being the wave vector satisfying the Bragg condition, d being a period of the structure, n_1 being a normalizing constant, and ϕ_0 being a constant, and $\phi_2(x)$ is defined in a similar way: $\phi_2(x)=n_2\sin(k_Bx+\phi_0)$ (see, e.g., the example Ref. [6]). We find $v_{\text{st}}=(c/2)(n_1n_2/n)$, where the effective index of refraction is $n=\omega/(ck_B)$. For $n_1n_2=n$ we arrive at the result $v_{\text{st}}=c/2$, discovered earlier through numerical experiments in Ref. [11].

The result of Eq. (8) explains also another feature of gap solitons observed in Ref. [11], viz., the instability of the steady state soliton solutions. Indeed, if solitons exist, they *must move* either in positive or negative directions with the velocity v_{st} (though the expansion is provided near the stationary waves). The actual direction is determined by the asymmetry of the problem, which can be introduced, for instance, through the boundary conditions.

In order to define b_l for $l\neq 1,2$ in the representation of Eq. (6) we multiply (1) by $\langle l|$ and retain terms proportional to μ^2; we obtain

$$b_l(x_1,x_2,\ldots;t_1,t_2,\ldots)=\frac{2c^2}{\omega_l^2-\omega^2}\left\{\langle l|\frac{\partial}{\partial x_0}|1\rangle\frac{\partial a_1}{\partial z}+\langle l|\frac{\partial}{\partial x_0}|2\rangle\frac{\partial a_2}{\partial z}\right\}. \tag{11}$$

The coefficients b_1 and b_2 are determined from the consideration of the third order of the expansion of Eq. (2) (note this is a mathematical peculiarity of the case at hand compared with [6,7]). This step, however, is different for different j. We start with the case $j\geqslant 3$. To this end we employ one more feature that follows from Eq. (8): i.e., the fact that we have only one dependent variable, for instance, a_1. To the third order of the expansion (i.e., to the terms proportional to μ^3), calculating the inner product with $\langle 1|$ and $\langle 2|$, and retaining harmonics with ω we obtain *two* nonlinear evolution equations

$$2i\frac{\partial a_1}{\partial t_2}+2iv_{\text{st}}\frac{\partial a_1}{\partial x_2}+\left(A_1-\frac{v_{\text{st}}^2}{\omega}\right)\frac{\partial^2 a_1}{\partial z^2}+(\lambda_{11}+i\lambda_{12})|a_1|^2a_1=-2v_{\text{st}}\frac{\partial b_2}{\partial x_1}-2i\frac{\partial b_1}{\partial t_1}, \tag{12a}$$

$$2i\frac{\partial a_2}{\partial t_2}+2iv_{\text{st}}\frac{\partial a_2}{\partial x_2}+\left(A_2-\frac{v_{\text{st}}^2}{\omega}\right)\frac{\partial^2 a_2}{\partial z^2}+(\lambda_{22}+i\lambda_{12})|a_2|^2a_2=2v_{\text{st}}\frac{\partial b_1}{\partial x_1}-2i\frac{\partial b_2}{\partial t_1}. \tag{12b}$$

Here

$$\lambda_{lj}=\frac{12\pi}{L}\int_0^L \chi^{(3)}(x)\phi_l(x)\phi_j(x)[\phi_1(x)\phi_1(x)+\phi_2(x)\phi_2(x)]dx \tag{13}$$

and

$$A_j=\frac{c^2}{\omega}\langle j|j\rangle+\frac{4c^2}{\omega}\sum_{l\neq 1,2}\frac{\langle j|\partial/\partial x_0|l\rangle\langle l|\partial/\partial x_0|j\rangle}{\omega_l^2-\omega^2}+i(-1)^{j-1}\frac{4c^2}{\omega}\sum_{l\neq 1,2}\frac{\langle j|\partial/\partial x_0|l\rangle\langle l|\partial/\partial x_0|m\rangle}{\omega_l^2-\omega^2}, \tag{14}$$

where $j\neq m$.

Let us first consider the case

$$\lambda_{11}=\lambda_{22}, \quad \lambda_{12}=0. \tag{15}$$

This condition is not satisfied by any nonlinearity.

Comparing Eq. (12a) and Eq. (12b) subject to the requirement (15) and taking into account the relation between a_1 and a_2 one immediately finds the compatibility conditions for the system. One of them determines b_1 and b_2 through the relation

$$b_1=ib_2=i\frac{A_2-A_1}{8v_{\rm st}}\frac{\partial a_1}{\partial z}. \tag{16}$$

Hence the companion term is orthogonal to the principal term (a discussion of this issue is given in the Appendix). This result is quite natural in the context of degenerate perturbation theory.

Solutions independent of x_2 are obtained through the equation for the amplitude a_1 that is written in the form of the NLS equation:

$$2i\frac{\partial a_1}{\partial t_2}+\Omega''\frac{\partial^2 a_1}{\partial z^2}+\lambda_{11}|a_1|^2a_1=0. \tag{17}$$

Thus *the effective dominant GVD* Ω'' *is originated by coupling and is the dispersion of the degenerate basic state* $\phi_1(x)+i\phi_2(x)$ (see Appendix). Using this last fact one finds that $\Omega^2=\frac{1}{2}(\omega_1^2+\omega_2^2)$ and, hence,

$$\Omega''=\frac{1}{2}(\omega_1''+\omega_2'')-\frac{v_{\rm st}^2}{\omega} \tag{18}$$

(here it is taken into account that $\omega_1'=\omega_2'=0$ at the Brillouin zone edge). This result is naturally coordinated with (A9), since in the presence of a gap $A_j=\omega_j''$ [see (14) and the expression for the GVD in the nondegenerate case derived in [6]]. It is to be pointed out that values ω_1'' and ω_2'' have different signs, and in some cases (such as, for example, weak modulation depth) have approximately equal moduli. Then one has a simple expression for the GVD: $\Omega''=-v_{\rm st}^2/\omega$. Thus in the mentioned cases the GVD induced by the coupling is negative.

For the existence of the coupled solitons of the above type it was necessary to have equal nonlinearity coefficients: $\lambda_{11}=\lambda_{22}$. Otherwise, if $\lambda=\lambda_{22}-\lambda_{11}\neq 0$, $\lambda_{12}=0$, the companion term does not decay with $|z|$ [see (19) below]. Since the multiscale expansion is valid only if the companion term is much less than the principal one we conclude that respective coupled gap solitons do not exist. The situation, however, is changed if $j=2$. In this case the effective nonlinearity associated with different edges is different but the distinction can be compensated by relatively stronger linear dispersion. Indeed, Eqs. (12a) and (12b) are now modified by adding $-\omega a_1$ and ωa_2 to their right hand sides, correspondingly. Equation (17) for the soliton amplitude is not changed but the companion term is now given by

$$b_1=ib_2 =i\frac{A_2-A_1}{8v_{\rm st}}\frac{\partial a_1}{\partial z}+\frac{i}{8v_{\rm st}}\int_c^z[\lambda|a_1(z')|^2-2\omega]a_1(z')dz' \tag{19}$$

instead of (16). The requirement for b_j to decay with $|z|$ determines the constant c in (19) and can be satisfied only under the condition

$$\int_{-\infty}^{\infty}(\lambda|a_1|^2-2\omega)a_1dz=0. \tag{20}$$

This last formula defines the soliton amplitude that cannot be arbitrary any more. In order to illustrate this fact let us consider the bright soliton of (17) ($\Omega''\lambda_{11}>0$),

$$a_1=2\alpha\frac{\exp(i\lambda_{11}\alpha^2t_2)}{\cosh(\sqrt{2\lambda_{11}/\Omega''}\alpha z)}, \tag{21}$$

where the constant α characterizes the amplitude. Then (20) results in $\alpha=\sqrt{\omega/\lambda}$. Hence, the coupled soliton amplitude is proportional to $\mu\sqrt{\omega}=\sqrt{\omega_2-\omega_1}$ [see (4)], i.e., the field intensity is proportional to the gap width.

IV. CONCLUSIONS

In this work we studied analytically properties of coupled gap solitons, i.e., solitons that originate in periodic dielectric media with a spectrum gap and are produced when linear modes corresponding to the lower and upper edge frequencies are coupled by nonlinearity. The coupled gap solitons may propagate with any frequency inside the gap. For our study we used the generalization of multiple scale expansion of de Sterke and Sipe [6,7] and analyzed the resulting low order wave equations that arise through this formalism. Our main assumption has been that the gap width is small compared with either the upper or lower gap edge. Our basic analytical finding is that coupled gap solitons are governed by the NLS equation. They are stable and propagate in the medium with a speed compared to half the speed of light. These results corroborate the numerical findings of [11] and show that the structures studied in [11] numerically may correspond to coupled gap solitons. A peculiarity of the gap soliton dynamics (compared with conventional gap solitons) is the influence of the type of nonlinear susceptibility on the characteristics of the excitation. In particular, the most important requirement for gap solitons to exist is $\lambda_{12}=0$ (note that this condition is provided by the constant nonlinear susceptibility $\chi^{(3)}(x)\equiv$const, and this can be considered as a perfect condition for existing coupled gap solitons). Otherwise periodic modulation of the nonlinearity causes resonant transformation of the energy between the mutually orthogonal states.

We belive that the results obtained bring more light on the problem of experimental excitation of gap solitons, showing how sensitive with respect to amplitude the phenomenon is.

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APPENDIX: GROUP VELOCITY DISPERSION

We generalize the arguments of Ref. [6] in order to interpret Ω''. Since the gap is considered small, we first analyze the resulting GVD as one occurring at a point compatible with a shrinking gap. Such a point corresponds to a doubly degenerate state of the spectrum. Hence one can apply a $\mathbf{k}\cdot\mathbf{p}$ method for degenerate states. We do this in two steps. First we introduce the normalized basis in which the perturbation

$$V_q=-c^2\left[2iq\left(\frac{\partial}{\partial x}+ik\right)-q^2\right]$$

is diagonal, and then we apply the well-known [15] expansion of the eigenvalues.

Let us introduce the notation $|j\}$ for the periodic part of the Bloch functions, i.e., $|j\rangle=e^{ikx}|j\}$. As is well known [15] one of the components of the basis in which the nondiagonal elements of V_q are zero is given by $c_+|1\}+c_-|2\}$, where

$$c_\pm=\left\{\pm\frac{\kappa}{2}\left[1\pm\frac{w_{11}-w_{22}}{W}\right]\right\}^{1/2}, \tag{A1}$$

with $\kappa=w_{12}/|w_{12}|$, $w_{ij}=\{i|V_q|j\}$, and $W=\sqrt{(w_{11}-w_{22})^2+4|w_{12}|^2}$.

Direct algebra yields

$$w_{12}=w_{21}=-2icq\langle 1|\partial/\partial x|2\rangle;\quad w_{jj}=c^2q^2\langle j|j\rangle. \tag{A2}$$

Since we are interested in the limit of small q and $w_{11}-w_{22}=O(q^2)$, the quantity W can be approximated by

$$W=4cq|\langle 1|\partial/\partial x|2\rangle|. \tag{A3}$$

Hence one can put

$$c_\pm=\left\{\pm\frac{1}{2}\left[1\pm\frac{qc(\langle 1|1\rangle-\langle 2|2\rangle)}{4|\langle 1|\partial/\partial x|2\rangle|}\right]\right\}^{1/2}. \tag{A4}$$

A direct consequence of this formula is that one of the basic states that make the operator V_q diagonal in the limit $q\to 0$ is given by $2^{-1/2}(|1\rangle+i|2\rangle)$. This gives an explanation of the result of Eq. (10): the gap soliton is an envelope modulating the states that diagonalize the operator V_q to the lowest order with respect to q. This also explains the form of the companion term since in the basis defined above the first order correction appears to be orthogonal (in the space of the eigenfunctions) to the term of the zero order [15].

The expansion of the eigenvalue Ω^2 corresponding to the eigenfunction $c_+|1\}+c_-|2\}$ is now given by

$$\Omega^2=\omega^2+\tilde{V}+\sum_{m\neq 1,2}\frac{|V_m|^2}{\omega^2-\omega_m^2}, \tag{A5}$$

where

$$\tilde{V}=(\{1|\overline{c_+}+\{2|\overline{c_-})V_q(c_+|1\}+c_-|2\})=2c^2q\langle 1|\partial/\partial x|2\rangle+\frac{q^2c^2}{2}(\langle 1|1\rangle+\langle 2|2\rangle)+O(q^3), \tag{A6}$$

$$|V_m|^2=|\{m|V_q(c_+|1\}+c_-|2\})|^2=c^2q^2[|\langle m|\partial/\partial x|1\rangle|^2+|\langle m|\partial/\partial x|2\rangle|^2+i\langle m|\partial/\partial x|1\rangle\overline{\langle m|\partial/\partial x|2\rangle}-i\overline{\langle m|\partial/\partial x|1\rangle}\langle m|\partial/\partial x|2\rangle], \tag{A7}$$

and the sum in (A5) excludes the states that are degenerated [in the terms of the initial statement of the problem they are $\phi_1(x)$ and $\phi_2(x)$, i.e., $m\neq 1,2$]. Comparing (A5) with the definition (9) one finds that

$$\Omega'=v_{\rm st} \tag{A8}$$

and

$$\Omega''=\frac{1}{2}(A_1+A_2)-\frac{v_{\rm st}^2}{\omega}. \tag{A9}$$

Thus Ω' is a group velocity associated with the mode $\phi_1(x)+i\phi_2(x)$, i.e., with the cw that is modulated, and $\Omega''=dv_{\rm st}/dk$ is its GVD.

[1] M. Sigalas, C. M. Soukoulis, E. N. Economou, C. T. Chan, and K. M. Ho, Phys. Rev. B **48**, 14 121 (1993), and references therein.
[2] H. G. Winful, Appl. Phys. Lett. **46**, 527 (1985).
[3] W. Chen and D. L. Mills, Phys. Rev. Lett. **58**, 160 (1987).
[4] D. L. Mills and S. E. Trullinger, Phys. Rev. B **36**, 947 (1987).
[5] J. E. Sipe and H. G. Winful, Opt. Lett. **13**, 132 (1988).
[6] C. M. de Sterke and J. E. Sipe, Phys. Rev. A **38**, 5149 (1988).
[7] C. M. de Sterke and J. E. Sipe, Phys. Rev. A **39**, 5163 (1989).
[8] D. N. Christodoulides and R. I. Joseph, Phys. Rev. Lett. **62**, 1746 (1989); A. B. Aceves and S. Wabnitz, Phys. Lett. A **141**, 37 (1989).
[9] C. M. de Sterke and J. E. Sipe, Phys. Rev. A **42**, 2858 (1990).
[10] J. E. Sipe, in *Guided Wave Nonlinear Optics*, edited by D. B. Ostrowsky and R. Reinisch (Kluwer Academic, Dordrecht, Netherlands, 1992).
[11] J. Peyraud and J. Coste, Phys. Rev. B **40**, 12 201 (1989).
[12] V. V. Konotop, Phys. Rev. A **51**, R3423 (1995).
[13] S. A. Bulgakov and M. Nieto-Vesperinas (private communication); see also D. R. Smith, R. Dalichaouch, N. Kroll, S. Schultz, S. L. Mc Call, and P. M. Platzman, J. Opt. Soc. Am. B **10**, 314 (1993).
[14] S. Lee and S.-T. Ho, Opt. Lett. **18**, 962 (1993).
[15] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon, New York, 1977).